Pressure Rise in a Liquid Fuel Tank Owing to Volumetric Heating

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Both the metallic casing and the liquid contents of a partially filled tank of a right-circular-cylinder configuration are subjected to a volumetric heating that is uniform in time and space, although the case and the liquid are heated at a different rate. The ullage includes the vapor of the liquid, and also a fixed amount of an inert gas, present for purposes of pressurization; the gases are taken not to be subject to heating. A theoretical analysis is undertaken to predict the increase in pressure in time as a consequence of the heating, which, for example, might arise from the absorption of an incident particle beam. The increase in pressure is caused by two effects: 1) the differential expansion of the liquid and casing as the temperature and pressure increase, and 2) the vaporization of some liquid to maintain saturation vapor pressure with an increase in temperature. The differential expansion is the mechanism for the large pressure rise for a small ullage; for a large ullage, the phase change is the more important mechanism, but the associated pressure rise is generally much smaller.

Nomenclature

a = inner radius of a right-circular-cylinder container
b = outer radius of a right-circular-cylinder container
c_j = heat capacity of phase j ($j = 1,2$)
c_p = heat capacity at constant pressure
$(c_p)_j$ = heat capacity at constant pressure of gaseous species ($j = i,v$)
c_v = heat capacity at constant volume
$h(t)$ = axial extent of the ullage in a right-circular cylinder
k_1 = thermal conductivity of the liquid
$L[T]$ = latent heat of vaporization at temperature T
L_0 = initial axial extent of the fluid-containing volume of the right-circular cylinder; also, $L[T_0]$
$m(t)$ = total gaseous mass in the ullage at time t
$\dot{m}(t)$ = time rate of change of $m(t)$
m_j = molecular weight of gaseous species j ($j = i,v$)
n = total number of moles of gas
$p(t)$ = pressure at time t
$p_i(t)$ = partial pressure of the inert gas in the ullage
$p_v(t)$ = partial pressure of the vapor in the ullage
$P_v[\theta]$ = saturation vapor pressure at temperature θ
Q_j = heat addition per mass per time to phase j
R = universal gas constant
R_j = R/m_j ($j = i,v$)
$T_j(t)$ = temperature at time t of phase j ($j = 1,2$)
t = time elapsed since onset of heating
u_1 = rate of regression of the liquid-gas interface in laboratory coordinates
$V(t)$ = fluid-containing volume at time t, gas plus liquid
$V_g(t)$ = gas-containing volume at time t
x_0 = $V_g(0)/V_0$
z = axial distance
γ = c_p/c_v
$\theta(t)$ = temperature of the gases in the ullage at time t
$\Theta(z,t)$ = deviation from the temperature of the bulk liquid at time t
κ_1 = $k_1/(\rho_1 c_1)$
$\rho_1(t)$ = density of the liquid
ρ_2 = density of the metallic casing
$\rho_i(t)$ = density of the inert gas in the ullage at time t
$\rho_v(t)$ = density of the vapor in the ullage at time t

Subscripts

i = inert gas of the ullage
v = vapor of the ullage
0 = value at time $t = 0$, or reference value
1 = liquid
2 = metallic casing

Introduction

WE seek to estimate the increase in time of the pressure in a partially filled liquid fuel tank in the configuration of a right-circular cylinder (Fig. 1). The pressure, spatially uniform to excellent approximation, rises because of a spatially and temporally uniform volumetric heating. The liquid contents are heated at a rate $\rho_1 Q_1$; the metallic casing is heated at a rate $\rho_2 Q_2$, where (for specificity) $Q_1 > Q_2$ but $Q_1/Q_2 = O(1)$. This form of heating approximates that, because of the absorption of an incident broad particle beam under unfocused operation, off from the peak of the so-called Bragg ionization curve.[1] For the cases to be examined, to satisfactory approximation, neither the vapor of the liquid, which is initially present at saturated levels, nor the inert gas also occupying the ullage, are taken to be heated. The particle-beam heating of a medium is related to the density of the medium. For constant-total-mass systems in which there is appreciable interphase mass transfer, such that the ullage becomes an appreciable fraction of the total volume, neglect of the heating of the gas relative to the liquid becomes a source of increasing error. For some conditions of possible interest, the gas density becomes 20% of the liquid density, or more, in extreme limits. Nevertheless, the convenient neglect of gas-phase heating is retained as justified in a wide range of interesting conditions. The mass of the inert gas, present for purposes of pressurization, always is invariant in time. It may be noted that much of the analysis to follow could be readily generalized to encompass temporally variant heating, but that generalization is not explicitly undertaken.

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The pressure rise in time results from two distinct contributions: 1) expansion of the liquid (in excess of the expansion of the metallic casing) under heating, such that the ullage is compressed into a smaller volume; and 2) phase transition of the liquid to vapor, such that saturation of the vapor is maintained and the pressure increases as a result of the increased vapor mass in an approximately fixed volume. Since approximate estimates are sought by simple analysis, only the dominant of the two mechanisms is considered in each of two cases. Specifically, it can be confirmed a posteriori that, for the small-ullage case, the pressure rise in time is primarily caused by expansion of the heated liquid, and phase transition contributes but a modest correction; for the large-ullage case, pressure rise is primarily caused by phase transition, and any expansion of the heated liquid (or of the ullage volume, for that matter) may be ignored. What is ignored as of negligible consequence in both cases is the possible dissociation of the vapor into a larger number of moles with an increase in temperature.

Implicit in the stratification of the phases depicted in Fig. 1 is that the heating occurs under normal gravity. For cases of interest here, the temperature rises in the solid and liquid to a few hundred degrees Kelvin above ambient over an interval of about 10 s. Thus, radiative heat transfer (e.g., from the solid to the surrounding air) is negligible, and free-convective boundary layers have limited opportunity to become established and convey heat from the solid to the air.[2,3] In addition, the ratio of radii (b/a) is such that any heat transfer from the outer portion of the metallic casing to ambient air has little consequence on the heat transferred from the casing to the contained liquid. There is only a small amount of heat transferred from the (normally hotter) casing to the liquid via a thin natural-convection boundary layer, and any such transfer results in slightly hotter fluid at the top of the liquid to counteract the local consequence of evaporative cooling.

Pressure Rise from Differential Expansion: The Small-Ullage Case

For the case in which the axial extent of the ullage is *much smaller* than the depth of the liquid in the tank, the differential expansion with the temperature rise of the liquid contents and of the metallic casing under heating results in a "crushing" of the ullage to a still smaller volume. This section is devoted to the approximate examination of this case under the convenient *idealizations* that 1) the temperature is spatially uniform (but temporally varying) in the gas and the liquid, and 2) no interphase mass transfer occurs.

The thermal diffusivity of most metals far exceeds that of most gases, and, in turn, the thermal diffusivity of most gases far exceeds that of most liquids; here, the thermal diffusivity of both liquid and gas is approximated as zero, such that the temperature of the gas, in general, need not be equal to the temperature of the liquid. One expects that the approximation of spatial uniformity may be more suitable for the liquid, which in fact, experiences axial thermal gradients over a quite thin (although growing) thermal boundary layer contiguous to the two-phase interface, than for the gas. The thermal boundary layers in the gas typically are thicker than for the liquid, and, as the volume of ullage is reduced, finite thermal gradients (owing to a near-top-wall thermal boundary layer and to a near-two-phase-interface thermal boundary layer) may arise across the entire gas phase. Temperature nonuniformity at the lateral interface between the liquid and solid is discussed subsequently.

If $p_i(t)$ denotes the pressure of the inert gas introduced for pressurization and $p_v(t)$ denotes the pressure of the vapor, then if both gases are taken to ideal gases,

$$p_i V_g = n_i R \theta \tag{1}$$

$$p_v V_g = n_v R \theta \tag{2}$$

where $n_j(t)$ is the number of moles of gas j. In this section, n_v as well as n_i are constant at their initial values. Following Dalton's law,

$$p = p_i + p_v, \qquad n = n_i + n_v \tag{3}$$

If subscript 0 denotes initial value, then

$$\frac{p(t)}{p_0} = \frac{\theta(t)}{\theta_0} \frac{x_0 V_0}{V_g(t)} \tag{4}$$

since the total number of moles is held constant in assessing the pressure rise owing to differential expansion of the liquid (relative to the metallic casing). It is taken that x_0, p_0, θ_0, and V_0 are given, where V_0 is the initial volume of the entire tank.

The initial state is in thermodynamic equilibrium, with the initial gas temperature θ_0 equal to the liquid temperature $T_1(0)$; therefore,

$$(p_v)_0 = P_v[\theta_0] \tag{5}$$

where P_v is the saturation vapor pressure of the liquid species, an empirically known functional of the temperature usually excellently represented[4] by the following (where, often, taking $C = 0$, $D = 0$ suffices):

$$P_v[\theta] = A\theta^D \exp[C\theta - (B/\theta)] \tag{6}$$

Thus, from θ_0, one finds $(p_v)_0$, whence $(n_v)_0$, and n_0. One may use Eqs. (5) and (6) at later times to suggest when condensation of vapor, ignored in the analysis, in fact, might be occurring.

At any time t, the volume of the container is given by [if $T_2(t)$ denotes the temperature in the solid]

$$V(t) \doteq V_0\{1 + \alpha_2[T_2(t) - T_0]\} \tag{7}$$

where the volume expansivity α_2 of the container is approximated as constant over the range of thermodynamic states of interest, and, by definition,

$$\alpha_2 = \frac{1}{V_0}\left(\frac{\partial V}{\partial T_2}\right)_p \tag{8}$$

The solid (as well as the liquid and the gas) is taken to be at the (known) temperature $T_0 = \theta_0$ initially. It is estimated that, for the pressure rise of interest here, the pressure-induced expansion of the volume of the container is negligible, and such a term does not appear in Eq. (8). In fact, $\alpha_2 \doteq 5 \times 10^{-5} K^{-1}$,

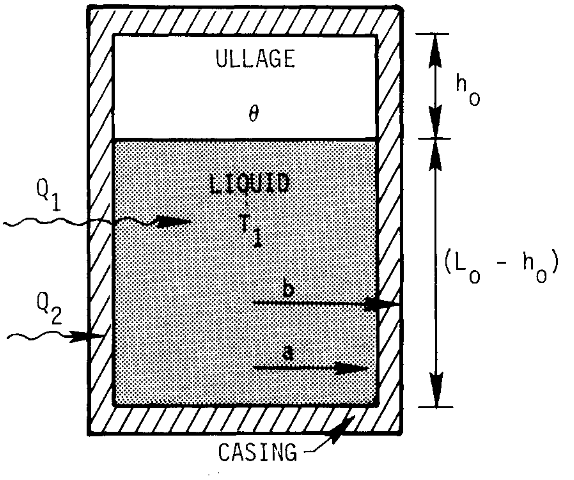

Fig. 1 Schematic of the cylindrical fuel tank. The ullage is not directly heated.

and thermal expansivity turns out to be responsible for merely a 0.5% increase in volume for cases of interest, a small correction for most purposes, but one that is retained here. Hence, it is necessary to solve for $T_2(t)$.

Because the temperature of the bulk of the solid casing T_2 is not equal to the temperature of the bulk of the liquid $T_1(<T_2)$, a *radially* diffusing thermal boundary layer exists at the liquid-solid interface. In time, this growing boundary layer vitiates the simplification of the spatially homogeneous core in the bulk of the liquid. A singular-perturbational treatment[5] of this phenomenon carefully establishes what is not readily evident from a separation-of-variables treatment, namely that the bulk of the fluid remains spatially homogeneous over the limited time interval of interest.

The expansion of the liquid under heating compresses the ullage, as opposed to expanding the container; therefore, if a denotes the (approximately constant) radius of the cylindrical tank, then

$$L_0 = V_0/(\pi a^2), \qquad h(t) = V_g(t)/(\pi a^2) \tag{9}$$

where L_0 is the initial height of the container, $h(t)$ the axial extent of the ullage with $h(0) = h_0$, and $[L_0 - h(t)]$ the approximate depth of the liquid. It follows that

$$x_0 = h_0/L_0 \tag{10a}$$

$$x_0 V_0/V_g(t) = h_0/h(t) \tag{10b}$$

The extent of the ullage at time t depends on the differential expansion of the solid and (the consequence of thermal and pressure effects) on the remaining liquid. If ρ denotes density, since subscript 1 refers to the liquid and subscript 2 refers to the solid casing,

$$h(t) \doteq L_0\{1 + \alpha_2[T_2(t) - T_0]\} - \frac{(L_0 - h_0)\rho_1[T_0,p_0]}{\rho_1[T_1(t),p(t)]} \tag{11a}$$

so, if m_v is the molecular weight of the vapor, the vapor density is

$$\rho_v(t) = \frac{m_v(n_v)_0}{\pi a^2 h(t)}, \qquad \rho_v(0) = (\rho_v)_0 \tag{11b}$$

In general, the density of the liquid is given by an equation of state of the form

$$\frac{\rho_1[T_1(t),p(t)]}{\rho_1[T_0,p_0]} = 1 - \alpha_1(T_0,p_0)[T_1(t) - T_0] + \beta_1(T_0,p_0)[p(t) - p_0] \tag{12}$$

where

$$\alpha_1 = -\frac{1}{\rho_1(T_0,p_0)}\left[\frac{\partial \rho_1(T_0,p_0)}{\partial T_1}\right]_p \tag{13}$$

$$\beta = \frac{1}{\rho_1(T_0,p_0)}\left[\frac{\partial \rho_1(T_0,p_0)}{\partial p}\right]_{T_1} \tag{14}$$

More generally, one uses $T_0 \to T_{\text{ref}}$, $p_0 \to P_{\text{ref}}$ in Eq. (12) because α_1 and β_1 may not be approximated adequately as constants over the range of states of interest.

To permit the solution of $p(t)$, one notes that the temperature of the bulk liquid $T_1(t)$ owing to the heating Q_1, and the temperature of the bulk solid $T_2(t)$ owing to the heating Q_2, are given by

$$T_1(t) = T_0 + (Q_1/c_1)t \tag{15a}$$

$$T_2(t) = T_0 + (Q_2/c_2)t \tag{15b}$$

where c_1 and c_2 are the (known) heat capacity of the liquid and of the solid, respectively, each approximated here as a constant over the range of temperature and pressure of interest. More generally, one could account for $c_1(T_1,p)$ and $c_2(T_2,p)$, but these variations tend to be modest. In writing Eq. (15a), one adopts the standard excellent approximation that "heat added reversibly to liquids is manifested almost wholly as a gain in internal energy, irrespective of the associated changes in p and ρ" (Ref. 6, p. 57); it is recalled that the heat capacities at constant pressure and at constant volume generally are effectively indistinguishable for a liquid. The temperature of the gas is taken to be implied by the relation for adiabatic compression

$$\frac{p(t)}{p_0} = \left[\frac{h_0}{h(t)}\right]^\gamma \tag{16}$$

where γ is the ratio of heat capacities for the gas mixture. Explicitly, $\gamma = (c_p/c_v)$, where

$$c_p = [(c_p)_v n_v m_v + (c_p)_i n_i m_i]/(n_v m_v + n_i m_i) \tag{17a}$$

$$c_v = c_p - Rn/[n_v m_v + n_i m_i] \tag{17b}$$

and $(c_p)_v$ and $(c_p)_i$ are given functions of temperature only, approximated here as constants.

For the seven unknowns $h(t)$, $T_1(t)$, $T_2(t)$, $\theta(t)$, $p_v(t)$, $p_i(t)$, and $p(t)$, one has the seven algebraic relations Eqs. (1–4), (15a), (15b), and (16). In these equations, the time t serves as a parameter, and, in general, iteration is required for a solution to be evolved in view of the nonlinearity. The quantity $\rho_v(t)$ may be eliminated in terms of the listed unknowns by use of Eqs. (2) and (11b); $V_g(t)$ may be eliminated by use of Eq. (10); and an empirical relation (i.e., an equation of state) for $\rho_1[T_1,p]$ is implied to be given [see Eqs. (12–14)]. The following quantities defined at the onset of heating also are to be furnished: $h_0(\equiv x_0 L_0)$, $T_0(=\theta_0)$, and p_0, along, of course, with the specified parameters m_v, m_i, L_0, a, Q_1, Q_2, α_2, $(c_p)_v$, $(c_p)_i$, etc.

Finally, the liquid can so expand under heating that the volume occupied by the ullage is reduced to nil. The solution for the pressure rise is pursued, at most, only to this circumstance. Further heating of the liquid leads to a pressure rise of the liquid under almost constant-volume conditions until a critical pressure is reached and the container bursts, but this interval is not examined here. The heating rate of the liquid exceeds that of the container, but the heat capacities are typically such that the temperature of the container exceeds that of the liquid; hence, under high heating rates, the onset of vapor bubbles and incipient boiling should be considered, but is not discussed further here.

Presentation of a few numerical examples is deferred until after the formulation of the pressure-rise problem for the large-ullage case in the next section.

Pressure Rise from Interphase Mass Transfer: The Large-Ullage Case

For the case in which the axial extent of the ullage is comparable to the depth of the liquid in the tank, vaporization of the liquid under heating results in an increased mass of gas in the ullage with time, with little increase of the ullage volume. The pressure rise in time now is sought for this case, in which provision for the spatial (specifically, axial) gradient of the temperature in the liquid is retained. This procedure is appropriate because the small thermal diffusivity of the liquid implies that the latent heat of phase transition, associated with vaporization of the liquid, is extracted from a thin layer of liquid contiguous to the two-phase interface. The thermal boundary layer thickens in time as $(\kappa_1 t)^{1/2}$, where κ_1 is the thermal diffusivity of the liquid. However, the time interval of interest is limited so that distances of $O[(\kappa_1 t)^{1/2}]$ are appreciably smaller than the depth of the liquid phase, conveniently taken to be of semi-infinite extent.

Just as corrections owing to spatial gradients and interphase mass transfer were ignored for convenience in the previous section, in which the pressure rise was primarily caused by changes of liquid properties with thermodynamic state, analogously, changes in liquid properties with thermodynamic state are neglected in this section, since the pressure rise is primarily caused by interphase mass transfer.

Since the solid casing does not enter prominently into the discussion in this section, the subscript unity on the symbol for temperature in the liquid can be discarded: $T_1(z,t) \rightarrow T(z,t)$.

The coordinate system is chosen such that the plane $z = 0$ lies at the gas-liquid interface at all times $t \geq 0$. In this coordinate system, the equation for conservation of energy in the liquid phase, taken to occupy the domain $0 > z > -\infty$, is given by

$$\rho_1 c_1 \frac{\partial T}{\partial t} + c_1 \dot{m} \frac{\partial T}{\partial z} - k_1 \frac{\partial^2 T}{\partial z^2} = \rho_1 Q_1 \tag{18}$$

where k_1 is the (constant) thermal conductivity of the liquid. The interphase mass transfer $\dot{m} = \dot{m}(t)$, where

$$\dot{m}(t) = \rho_1 u_1(t) = \rho_v(0,t) u_v(t) \tag{19}$$

On the other hand, the gas phase is confined to the domain $h(t) > z > 0$, where $h(0) = h_0 = x_0 L_0$. In fact,

$$h(t) = h_0 + \int_0^t u_1(t_1)\,\mathrm{d}t_1 \tag{20}$$

where the increase in ullage depth in time is $O(\rho_v/\rho_1)$, and a term of this magnitude is ignored in writing Eq. (18), so $h(t) = h_0$.

The thermodynamic state in the ullage is taken to be spatially uniform, and the mass of vapor in $h_0 > z > 0$ is given by

$$m(t) = h_0 \rho_v(t) \Rightarrow \dot{m}(t) = h_0 \dot{\rho}_v(t) \tag{21}$$

where the overdot denotes a total derivative with respect to time. Since the gas is taken to be saturated with vapor at all times (unless a contradiction arises),

$$p_v(t) = P_v[\theta(t)] = \rho_v(t) R_v \theta(t), \quad R_v = R/m_v \tag{22}$$

From Eqs. (1–3), it is evident that knowledge of $\theta(t)$ yields $p(t)$, since $\{p(0) \equiv p_0 = (p_i)_0 + P_v[\theta_0]\}$

$$p(t) = (p_i)_0\,(\theta(t)/\theta_0) + P_v[\theta(t)] \tag{23}$$

At the two-phase interface,[7,8] continuity of temperature holds at

$$T(0,t) = \theta(t) \tag{24}$$

Also, continuity of enthalpy flux at $z = 0$ is given by Eq. (25), if c_p is the heat capacity of the vapor (the subscript v used in the previous section is dropped in this section), and if L_0 (not to be confused with the tank height introduced in the previous section) is the latent heat of phase transition at a reference temperature T_0,

$$\begin{aligned} -k_1 \frac{\partial T(0,t)}{\partial z} &+ \dot{m}(t) c_1 [T(0,t) - T_0] \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \{m(t) c_p [\theta(t) - T_0] + m(t) L_0\} \end{aligned} \tag{25}$$

or, by Eq. (24),

$$-k_1 \frac{\partial T(0,t)}{\partial z} = c_p m(t) \frac{\partial T(0,t)}{\partial t} + \dot{m}(t) L[T(0,t)] \tag{26}$$

Here

$$L[T(0,t)] = L_0 + (c_p - c_1)[T(0,t) - T_0] \tag{27}$$

where, in general, $c_1 > c_p$, so $L[T(0,t)] < L_0$ for $T(0,t) > T_0$. In fact, caution is necessary so that the analysis is not extended beyond the triple point, where distinction of the gas and liquid does not exist. From Eqs. (21–24),

$$m(t) = \frac{h}{R_v} \frac{P_v[T(0,t)]}{T(0,t)} \tag{28}$$

$$\dot{m}(t) = \frac{h}{R_v} \left[\frac{\mathrm{d}}{\mathrm{d}T} \left\{ \frac{P_v[T(0,t)]}{T(0,t)} \right\} \right] \frac{\partial T(0,t)}{\partial t} \tag{29}$$

From Eqs. (27–29), Eq. (26) is a constraint on $T(0,t)$, albeit a highly nonlinear one.

The boundary/initial-value problem for $T(z,t)$ is given by Eqs. (18) and (26), and the following two constraints:

$$T(-\infty,t) \rightarrow T_0 + (Q_1/c_1)t \tag{30}$$

$$T(z,0) = T_0 \tag{31}$$

That is, the reference temperature for the enthalpy is the (uniform) initial temperature.

The problem may be rephrased usefully in terms of $\Theta(z,t)$, where

$$T(z,t) = T_0 + (Q_1/c_1)t + \Theta(z,t) \tag{32}$$

so that, if $\kappa_1 = k_1/(\rho_1 c_1)$,

$$\frac{\partial \Theta}{\partial t} + \frac{\dot{m}}{\rho_1} \frac{\partial \Theta}{\partial z} - \kappa_1 \frac{\partial^2 \Theta}{\partial z^2} = 0, \quad 0 > z > -\infty \tag{33}$$

$$\Theta(-\infty,t) = 0, \; \Theta(z,0) = 0 \tag{34}$$

$$\begin{aligned} -k_1 \frac{\partial \Theta(0,t)}{\partial z} &= c_p \left\{ m \left[T_0 + \frac{Q_1}{c_1} t + \Theta(0,t) \right] \right\} \left[\frac{Q_1}{c_1} + \frac{\partial \Theta(0,t)}{\partial t} \right] \\ &+ \left\{ \dot{m} \left[T_0 + \frac{Q_1}{c_1} t + \Theta(0,t) \right] \right\} \left\{ L \left[T_0 + \frac{Q_1}{c_1} t + \Theta(0,t) \right] \right\} \end{aligned} \tag{35}$$

The notation is such that

$$m(t) = m[T(t)] = m[T_0 + (Q_1/c_1)t + \Theta(0,t)] \tag{36}$$

and similarly for $\dot{m}(t)$, with the obvious addition that

$$\frac{\partial T(0,t)}{\partial t} = \frac{Q_1}{c_1} + \frac{\partial \Theta(0,t)}{\partial t} \tag{37}$$

The nonlinear term[9] in Eq. (33) is anticipated to be small relative to the other two terms, i.e.,

$$\left(\frac{\dot{m}}{\rho_1} \right) \left(\frac{\partial \Theta}{\partial z} \right) \Big/ \left(\frac{\partial \Theta}{\partial t} \right) = O\left(\frac{\rho_v}{\rho_1} \right) \tag{38}$$

where $(\rho_v/\rho_1) = \varepsilon \ll 1$. Hence, it is adopted that

$$\frac{\partial \Theta}{\partial t} - \kappa_1 \frac{\partial^2 \Theta}{\partial z^2} \doteq 0 \tag{39}$$

and the smallness of the omitted term is confirmed a posteriori.

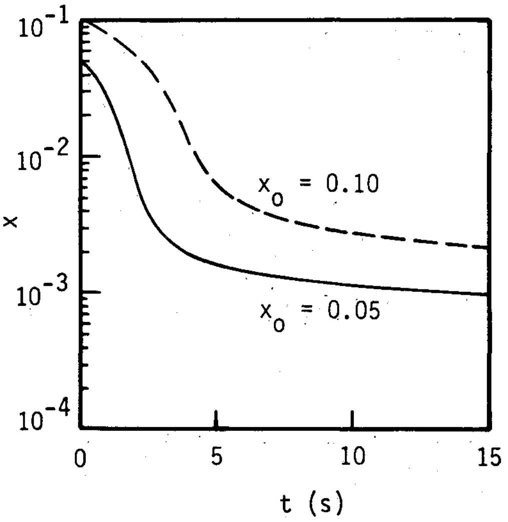

Fig. 2 Ullage fraction, $x(t)$, for the small-ullage-case parameters.

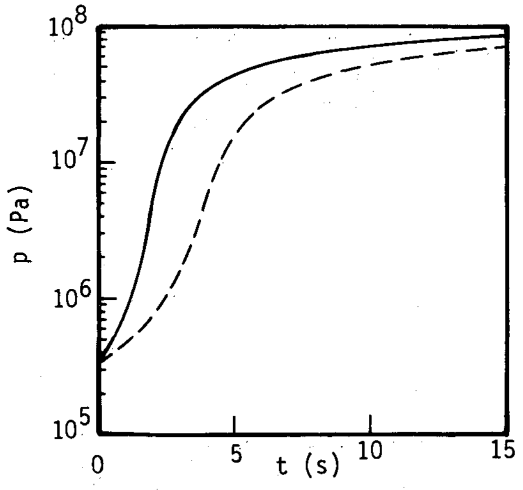

Fig. 3 Total pressure, $p(t)$, associated with the cases presented in Fig. 2.

Equation (35) is rewritten in the shorthand

$$-k_1\frac{\partial\Theta(0,t)}{\partial z}=f\left[t,\Theta(0,t),\frac{\partial\Theta}{\partial t}(0,t)\right] \tag{40}$$

From Carslaw and Jaeger,[7]

$$-\Theta(0,t)=\frac{\kappa_1^{1/2}}{k_1\pi^{1/2}}\int_0^t\frac{f\left[x,\Theta(0,x),\dfrac{\partial\Theta(0,x)}{\partial x}\right]}{(t-x)^{1/2}}\mathrm{d}x \tag{41}$$

This Volterra integral equation is amenable to numerical solution by Picard iteration ($n=0,1,2,3,\ldots$), although convergence is obtained for only small enough heat transfer at the interface for a short enough interval of time

$$-\Theta^{(n+1)}(0,t)=\frac{\kappa_1^{1/2}}{k_1\pi^{1/2}}\int_0^t\frac{f[x,\Theta^{(n)}(0,x),\Theta_x^{(n)}(0,x)]}{(t-x)^{1/2}}\mathrm{d}x \tag{42}$$

where subscript x denotes partial differentiation. The choice $\Theta^{(0)}(0,x)=0$ is convenient; for this trial, with the saturation vapor pressure in the form of Eq. (6) with $C=D=0$,

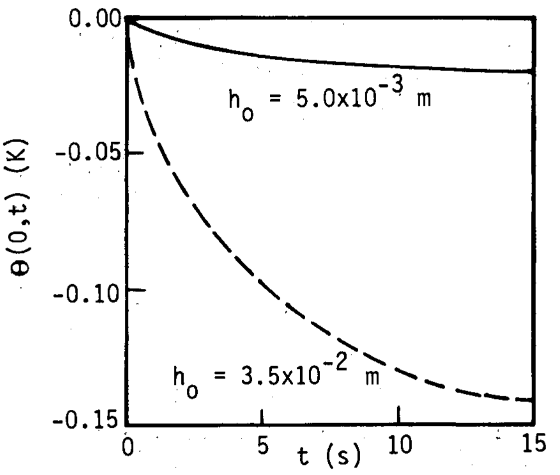

Fig. 4 Interfacial deviation temperature, $H(0,t)$, for the large-ullage-case parameters.

$$f[x,0,0]=c_p\frac{hQ_1}{R_v c_1}\frac{A\exp\{-B/[T_0+Q_1(x/c_1)]\}}{[T_0+Q_1(x/c_1)]^2}\times\left\{1+\frac{L_0+(c_p-c_1)Q_1(x/c_1)}{c_p[T_0+Q_1(x/c_1)]^2}\left[B-\left(T_0+Q_1(x/c_1)\right)\right]\right\} \tag{43}$$

The quantity $\Theta_x^{(n)}(0,0)$ may be assigned by extrapolation from values at finite time.

Numerical Examples

In all cases to be examined, the casing is taken to be of aluminum, the liquid is taken to be unsymmetrical dimethyl hydrazine (UDMH) and the gas present for pressurization is taken to be nitrogen. The pertinent thermodynamic properties of these substances are available[7,10] and will not be enumerated explicitly below. The uniform initial temperature $T_0=300$ K and the initial total pressure $p_0=0.345$ MPa; the initial (saturation) vapor pressure of UDMH $(P_v)_0=P_v[T_0]\doteq 26.0$ kPa.

First, exemplary computations are presented for the small-ullage case discussed earlier (Figs. 2 and 3). In these computations, the heating rate of the liquid $Q_1=1.25\times10^4$ J/(kg-s), the heating rate of the casing $Q_2=8.75\times10^3$ J/(kg-s), and the ratio of heat capacities $\gamma=1.4$. The initial volume $V_0=1.5\times10^{-5}$ m^3, and the fraction of the volume that initially is ullage x_0 is taken to be either 0.05 or 0.10. The small volume is adopted to correspond to experiments undertaken at an available particle-beam heating facility; the results of the experiments proved consistent with the results to be reported. Although results are reported for a 15-s heating interval (because the melting point of aluminum is achieved at around 16 s), the calculations are of significance only until about 10 s for several reasons: the data for the density of UDMH as a function of temperature and pressure are not incorporated adequately beyond that time, the aluminum begins to soften (so that the simple treatment of expansivity is suspect), and the critical point of UDMH is being approached (so that the heating of the gas may no longer be neglected). In accord with anticipation, the rise of pressure as a function of time is significant for the small ullage, with ullage fraction at time t in Fig. 2 referring to the ratio of the volume containing gas at time t to the total tank volume at time t.

If, in this small-ullage case, one dispenses with the model of adiabatic compression for the gas adopted in Eq. (16), and if, instead, one takes the temperature of the ullage (postulated to be saturated with vapor at all times) to be equal to the temperature of the liquid, so that the entire contents of the tank are at

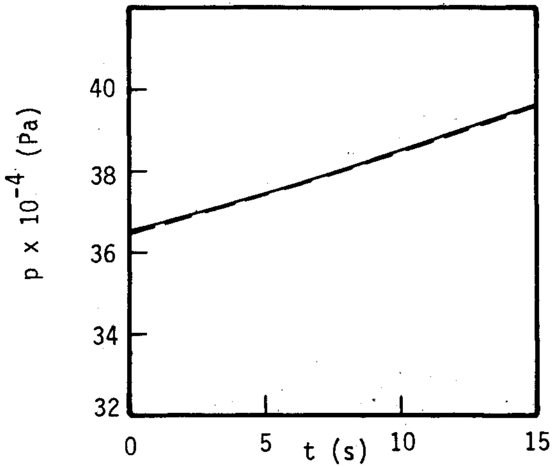

Fig. 5 Total pressure, $p(t)$, associated with the cases presented in Fig. 4.

one temperature at any time, then, in a calculation that accounts for the differential expansion of the liquid and of the casing, one obtains a pressure rise in time effectively identical to that presented in Fig. 3. In this alternate approximation, an attempt is made to account for the removal of an amount of latent heat from the liquid consistent with the interphase mass transfer; since the treatment is spatially homogeneous, this heat of phase transition is extracted uniformly from the entire mass of remaining liquid.

Next, exemplary computations are presented for the large-ullage case discussed earlier (Figs. 4 and 5). The constant properties adopted for the liquid and the gas are as follows: $\rho_l = 7.86$ kg/m^3, $c_l = 7.04 \times 10^2$ J/(kg-s), $k_l = 3.76 \times 10^{-2}$ J/(m-s-K), $c_p = 3.00 \times 10^2$ J/(kg-s), and $L_0 = 1.30 \times 10^5$ J/kg. The heating rate of the liquid $Q_l = 6.00 \times 10^3$ J/(kg-s), and the initial depth of the ullage h_0 is taken to be either 0.005 m or 0.035 m. A sevenfold increase in h_0 has no significant effect on results. The temperature at the two-phase interface is altered very modestly from the temperature in the bulk of the liquid, therefore, results obtained from a very simple equilibrium solution, with the temperature at the interface determined from volumetric heating, have been shown to suffice. While the heating rate of the liquid has been reduced to half that for the small-ullage case, still the pressure rise is roughly 10% of the initial pressure in the large-ullage case, whereas it is over 200 times the initial pressure in the small-ullage case over the same 15-s heating interval.

Acknowledgments

The authors are grateful to David Lewis for suggesting the problem. They are indebted to George Carrier for several discussions of the problem. They also wish to thank Janet Hixson for preparation of the manuscript and Asenatha McCauley for preparation of the figures.

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